Project 2: Koch Snowflakes and Fractals

Douglas Meade, Ronda Sanders, and Xian Wu
Department of Mathematics

Overview

The word "fractal" is often used in referring any object that is recursively constructed so that it appears similar at all scales of magnification. There are many examples of complex real-life phenomena, such as chaos, ferns, mountains, river networks, biological growth, that can be described and studied using fractals. In this lab and project, we will analyze and generate a classic fractal, the Koch snowflake, and its variations. While it is natural to use a computer to do recursive constructions, we will focus on applications of sequences and series in our study.

A Variation of Koch Snowflake

1. **Basic Construction:** While the classic Koch snowflake starts with an equilateral triangle, this variation starts with a square. A smaller square is then added to each of the four sides. It is done in such a way that a side of each new square is the middle one-third of each side of the original figure. This process is repeated again and again in each successive iteration. The final snowflake is the limit of this construction. To see the first five levels of such a construction, check the course webpage:

   `http://www.math.sc.edu/calclab/142L-F09/labs/`→ KochSquare

   We will use Maple to find the area and perimeter of this snowflake for the first five levels and then develop general forms so we can find the total area and perimeter of the final snowflake.

2. **Working with Maple:**

   (a) **Note:** You should always `restart` from the beginning after any modification.
   ```
   > restart;
   ```
   (b) **Initial Results (Level 0):** The following should be clear, since Level 0 is a square with side length 1.
   ```
   > LenSide[0] := 1;
   > NumSide[0] := 4;
   > TotPerim[0] := NumSide[0]*LenSide[0];
   > TotArea[0] := LenSide[0]*LenSide[0];
   ```
   (c) **Results up to Level 5 and limits:** We are going to use a typical for-loop to generate the results through Level 5. The commands are similar to those introduced in Lab J. Please pay close attention as your TA works out recursive formulas for TotPerim[n] (a sequence) and TotArea[n] (a series) by hand.
   ```
   > n:=5:
   > for k from 1 to n do
   >   level:=k;
   >   LenSide[k] := LenSide[k-1]/3.0;
   >   NumSide[k] := 5*NumSide[k-1];
   >   TotPerim[k] := NumSide[k]*LenSide[k];
   >   TotArea[k] := TotArea[k-1]+NumSide[k-1]*TotArea[0]/9.0^k;
   > end do;
   ```

3. **Developing General Forms for Perimeter and Area**
   (a) We will include a second `restart` command to reset our variables.
   ```
   > restart;
   ```
   (b) First notice that we are getting a sequence of flakes with more and more sides by the same construction: each side is replaced by five sides of one-third of its length at the next stage. Therefore, as we started at Level 0 with 4 sides of length 1, there are $4 * 5^n$ sides of length $(1/3)^n$ at stage n. That gives the formula
   $$P_n = 4 * 5^n * (1/3)^n = 4 * (5/3)^n$$
   for the perimeter of the flake at stage n. We can then use Maple to find the perimeter for Levels 1-5 and the total perimeter of the final flake.
   ```
   > P:= n-> 4*(5/3)^n;
   > evalf(seq(P(n), n=1..5));
   > limit(P(n), n=infinity);
   ```
   This sequence diverges and the perimeter of the final flake is hence infinite.
   (c) To get a formula for the area, notice that the new flake at stage $n \geq 1$ is obtained by adding a square of side length $(1/3)^n$ to each side of the previous flake. Therefore, the additional area added at Level $n \geq 1$ is:
   $$\text{Additional Area} = 4 * 5^{n-1} * (1/3)^n * (1/3)^n = (4/5) * (5/9)^n.$$
   Since the area at Level 0 is 1 and the above area is added at each stage thereafter, we hence obtain the following series for the area of the flake at stage $n \geq 1$:
   $$A_n = 1 + \sum_{k=1}^{n} (4/5) * (5/9)^k.$$
   We can then use Maple to find the area for Levels 1-5 and the total area of the final flake.
   ```
   > A:= n-> 1+sum((4/5)*(5/9)^k, k=1..n);
   > evalf(seq(A(n), n=1..5));
   > limit(A(n), n=infinity);
   ```
   Notice that the general form for the area is a geometric series, so if we found the limit using our knowledge of geometric series, we would know that this series converges to
   $$1 + \frac{4/9}{1-(5/9)} = 1 + 1 = 2.$$

Project 2: Koch Snowflakes Your report should follow the guidelines set forth in the Project Report Guidelines on the lab website and is due by the date specified by your TA. It should cover the following:

On the course webpage, you will find Maple worksheets that construct the classic Koch snowflake and a variation of the classic flake.

`http://www.math.sc.edu/calclab/142L-F09/labs/`→ ClassicKoch and KochVariation

For each of the given flakes, you should:

1. Present details to show that you fully understand the given flake construction. The given Maple worksheet should be used as a tool to help you verify and visualize.
2. Find the area and perimeter up to Level 5.
3. Develop general forms for the perimeter and area. (You should, of course, include these general forms in your report.)
4. Use Maple to discover the perimeter and area of the final flake, which is the limit of each construction.
5. Have a discussion on what you have learned, such as some of the interesting properties of the flakes and how your knowledge of sequences and series aided your evaluations.