Multilevel Preconditioning for Partition of Unity Methods - Some Analytic Concepts *

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Abstract

This paper is concerned with the construction and analysis of multilevel Schwarz preconditioners for partition of unity methods applied to elliptic problems. We show under which conditions on a given multilevel partition of unity hierarchy (MPUM) one even obtains uniformly bounded condition numbers and how to realize such requirements. The main anlytical tools are certain norm equivalences based on two-level splits providing frames that are stable under taking subsets.

Key Words: Multilevel partition of unity methods, two-level splits, multilevel expansions, additive Schwarz preconditioner, smoothness spaces, local linear independence, stability, Besov spaces

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1 Introduction

The so called *meshless methods* are drawing increasing attention in many areas of engineering applications since they avoid notorious difficulties with meshing complicated domains, in particular, when dealing with three or more spatial variables. Meshless methods have come under various names such as "moving least squares", "partition of unity method (PUM)", "radial basis functions", "web splines", "generalized finite elements" or " smoothed particle hydrodynamics". A recent account of the state of the art can be found in [1], see also the references cited there. There are close conceptual links with more theoretically motivated directions of studies in the group of Triebel (see e.g. [11]) centering on atomic decompositions related to PUM. While most of the numerical work refers to issues like error estimates and functionality of the method, comparatively less seems to be known about fast solution methods for the systems of equations arising from meshless discretization concepts. There is an impressive body of work on *multigrid solvers* for certain variants of PUM documented in [7, 15, 9] which shows very good performance. On the other hand, it seems that rigorous

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estimates are still lacking nor is it clear how well these techniques comply with adaptive strategies.

Here we shall focus on the following model problem. Let $a(\cdot,\cdot) : V \times V$ be a symmetric bilinear form on a Hilbert space V with norm $\|\cdot\|_V = \langle\cdot,\cdot\rangle^{1/2}$ that is V-elliptic, i.e. there exist positive constants c_a, C_a such that

$$a(v,v) \geq c_a\|v\|_V^2, \quad |a(v,w)| \leq C_a\|v\|_V\|w\|_V, \quad v,w \in V. \tag{1.1}$$

For any given $f \in V'$ find $u \in V$ such that $a(u,v) = \langle f, v\rangle$, $v \in V$. In what follows V will always be assumed to be one of the spaces $H^1(\Omega)$ or $H_0^1(\Omega)$ corresponding to Neumann or Dirichlet boundary conditions. We shall always assume in what follows that Ω is a bounded *extension* domain. This means that Ω has a sufficiently regular boundary to permit any element v of any Sobolev or Besov space $X(\Omega)$ over Ω to be extended to $\tilde{v} \in X(\mathbb{R}^d)$, $\tilde{v}|_\Omega = v$, in such a way that $\|v\|_{X(\mathbb{R}^d)} \leq C_X\|v\|_{X(\Omega)}$. This is, for instance, the case when the boundary of Ω is piecewise smooth and a uniform cone condition holds for Ω.

The objective of this paper is to develop a multilevel Schwarz preconditioner in the PUM setting that provides even uniformly bounded condition numbers for elliptic boundary value problems. The primary focus of this investigation is a sound theoretical foundation of this issue. Our emphasis here is on bringing out some basic principles that seem to be relevant in such a context and most of the results will be asymptotic in nature. Moreover, it will be seen to comply well with adaptive refinements. Many quantitative aspects such as treating inhomogeneous boundary conditions, dealing with jumping diffusion coefficients or the important issue of quadrature will not be addressed here.

In Section 2 we shall describe the general setting of a multilevel covers of Ω on which the construction of multilevel systems of atoms and resulting partition of unity hierarchies (MPUH) in Section 3 will be based upon. The central issues in this section are to establish certain scalewise stability properties as well as approximation bounds. The latter estimates as well as certain multilevel representations are based on suitable versions of quasi-interpolants. In particular, we shall identify several conditions, especially concerning certain *local linear independence properties*, that, combined with two-level splits in multilevel expansions, will later be crucial for proving norm equivalences based on these representations in many smoothness spaces. In Section 4 we return to the problem (1.1) and formulate a multilevel Schwarz preconditioner based on the multilevel representations from the previous section. Moreover, we indicate some possible combination with adaptive solution strategies as well as the relevance of best N-term approximation in this context. The fact that the proposed preconditioner gives rise to uniformly bounded condition numbers is a consequence of the norm equivalences established in Section 5. There some effort is spent on proving these norm equivalences for the full range of smoothness spaces in $L_p(\Omega)$ for $0 < p \leq \infty$. While the stable splittings needed in Section 4 only require Sobolev spaces in $L_2(\Omega)$, dealing with best N-term approximation requires working, in particular, with the case $p < 1$.

For the sake of convenience we shall sometimes use the notation $a \lesssim b$ if $a \leq Cb$ with an absolute constant C independent of all parameters on which a, b depend. Similarly, $a \sim b$ means that both relations $a \lesssim b$ and $b \lesssim a$ hold.

2 Discrete Multilevel Covers of $\Omega \subseteq \mathbb{R}^d$

We wish to discretize (1.1) with the aid of a multilevel partition of unity hierarchy (MPUH) which will be based on certain multilevel covers of the domain Ω. To this end, let $B_r(x)$ denote the (open) ball of radius $r > 0$ and center $x \in \mathbb{R}^d$. We call an open set $\theta \subset \mathbb{R}^d$ a *proper cell* if it has the following properties:

(p1) θ is *star-shaped*, i.e. there exists a "center" x_θ such that for any $x \in \partial\theta$ (the boundary of θ) the line segment $[x_\theta, x]$ connecting x and x_θ is contained in $\bar{\theta}$.

(p2) One can find $r_1 \leq r_2$ such that for a given $R > 0$

$$B_{r_1}(x_\theta) \subseteq \theta \subseteq B_{r_2}(x_\theta), \quad \text{where} \quad r_2/r_1 \leq R.$$

Clearly, balls as well as hypercubes are proper cells. Note that proper cells can be dilated. For any positive a let

$$s_a(\theta) := \{x \in \mathbb{R}^d : \exists\, y \in \partial\theta \ \text{ s.t. } \ x \in [x_\theta, x_\theta + a(y - x_\theta)]\}. \tag{2.2}$$

For a given compact domain $\Omega \subset \mathbb{R}^d$ (with the properties mentioned in the previous section) or $\Omega = \mathbb{R}^d$, we assume that Θ is a discrete multilevel collection of proper cells in $\mathbb{R}^d$ ($d \geq 1$) of the form

$$\Theta = \bigcup_{m=0}^{\infty} \Theta_m$$

with the following properties: For given positive constants $a_0, a_1, a_2, \ldots$ and N_1 one has:

(C1) For $m \in \mathbb{N}_0$ we have $\Omega \subseteq \bigcup_{\theta \in \Theta_m} \theta$ and $a_1 2^{-a_0 m} \leq |\theta| \leq a_2 2^{-a_0 m}$ for all $\theta \in \Theta_m$, where $|\theta|$ denotes the volume of θ.

(C2) At most N_1 cells from Θ_m may have a nonempty intersection.

(C3) If $\theta \cap \theta' \neq \emptyset$, $\theta, \theta' \in \Theta_m$, then $|\theta \cap \Omega| \geq a_3|\theta|$ and $|(\theta \setminus \theta') \cap \Omega| \geq a_3|\theta|$.

(C4) For every $x \in \Omega$ and $m \in \mathbb{N}_0$ there exists $\theta \in \Theta_m$ such that $x \in s_{a_4}(\theta)$ for some $a_4 < 1$.

(C5) For all $\theta \in \Theta_m$, $\eta \in \Theta_{m+1}$ we either have $\theta \cap \eta = \emptyset$ or $|\theta \cap \eta \cap \Omega| > a_5|\eta|$.

On account of (C3) and (C5) we shall from now on adopt the convention that θ is always understood to mean $\theta \cap \Omega$.

With any cover of the above type we can associate a parameter vector $\mathbf{p} = \mathbf{p}(\Theta)$ containing all the constants appearing in the above requirements. Note that by (C2) the number of overlaps is controlled, while (C3) says that every two cells from Θ_m are essentially different. (C4) means that every point in the domain is "well covered" by at least one proper cell, while (C5) controls the overlap between cells from two successive levels. Somewhat more can be said.

Remark 2.1. *From the definition of a proper cell and (C1) it follows that for any* $\theta \in \Theta_m$ *we have* $\operatorname{diam} \theta \sim 2^{-a_0 m/d}$ *with constants of equivalence depending only on* $\mathbf{p}(\Theta)$*. Moreover, for any* $\theta \in \Theta_m$ *and* $\theta' \in \Theta_{m+1}$ *there exist balls*

$$B_{r_1}(x_{\theta'}) \subseteq \theta', \quad \theta \subseteq B_{r_2}(x_\theta), \quad s.t. \;\; r_2/r_1 \leq a_6,$$

with a_6 *depending only on* $\mathbf{p}(\Theta)$*.*

Of course, thinking of applications where the centers x_θ are given, depending on their distribution, it might be difficult to construct covers with the above properties. When thinking of applications to boundary value problems, one is free to choose centers as well as the shape of cells that accommodate the construction and covers. Note that one typically does not adapt the covers to domain boundaries. The perhaps simplest construction can be sketched as follows. For simplicity let $\Omega = \mathbb{R}^2$ and let the lattice points $k = (k_1, k_2) \in \mathbb{Z}^2$ be the centers at level 0. Let

$$\Theta_m = \{2^{-m}[k_1 - b, k_1 + b] \times [k_2 - b, k_2 + b] := 2^{-m}(k + [-b, b]^2) : k \in \mathbb{Z}^2\}, \tag{2.3}$$

where $b \in (1/2, 1)$ is fixed. Thus $a_0 = 2 = d$, $|\theta| = 2^{-2m}(2b)^2$ for $\theta \in \Theta_m$, and obviously, for $\theta, \theta' \in \Theta_m$, $\theta \cap \theta' \neq \emptyset$ one has $|\theta \cap \theta'| \geq 2^{-2m}(2b-1)^2$. Likewise when $\theta' \in \Theta_{m+1}$, $\theta \in \Theta_m$ have nonempty intersection, one can verify that

$$|\theta \cap \theta'| \geq \begin{cases} 2^{-2m}\left(\frac{3b}{2} - 1\right)^2 & \text{if} \;\; 2/3 < b < 1; \\ 2^{-2m}\left(\frac{3b}{2} - \frac{1}{2}\right)^2 & \text{if} \;\; 1/2 < b \leq 2/3. \end{cases} \tag{2.4}$$

Hence, one has $a_1 = a_2 = (2b)^2$ in (C1), $N_1 = 4$ in (C2). Moreover, note that $|\theta \cap \theta'| \geq (2b - 1/2b)^2 |\theta|$, $a_4 = 1/2b$ in (C4), and in (C5) $a_5 = (\frac{3}{2} - \frac{1}{b})^2$ when $b > 2/3$, while $a_5 = (\frac{3}{2} - \frac{1}{2b})^2$ when $1/2 < b \leq 2/3$. Of course, rescalings may be necessary near domain boundaries.

Note that when $b \leq 2/3$ certain intersections of small cells with cells from the previous level in (C5) become empty which accounts for the two cases in (2.4). It is also clear how to extend this to general $d \geq 3$.

Remark 2.2. *The above example has an additional property that will be exploited later, namely,*

$$\forall\, \theta \in \Theta_m \;\; \exists\, \Omega_\theta \subset \theta \;\; s.t. \;\; \theta' \cap \Omega_\theta = \emptyset, \; \forall\, \theta' \in \Theta_m \setminus \{\theta\} \tag{2.5}$$

and

$$|\Omega_\theta| \geq a_6 |\theta|. \tag{2.6}$$

We shall refer to a cover with this property as a sparse *cover and* a_6 *will be added to the parameter list* $\mathbf{p}(\Theta)$*. In the above example we have* $a_6 = (1-b)^2$*.*

An important point about covers of the above type is that the spatial localization offered by moving to higher levels is *isotropic*. The setting presented here may be viewed as a specialization of a more general framework put forward in [3] which aims at capturing also anisotropic features.

Finally it will be convenient to confine the subsequent discussion to the slight further constraint that all proper cells are affine images

$$\theta = A_\theta(\overset{\circ}{\theta}) \tag{2.7}$$

of a single *reference cell* $\overset{\circ}{\theta}$ of volume $|\overset{\circ}{\theta}| \sim 1$. In the above example the A_θ are just compositions of shifts and dilations.

From now on we shall always assume that Θ satisfies properties (C1) – (C5) for some parameter vector $\mathbf{p}(\Theta)$ as well as that (2.7) holds.

3 Construction of Multilevel Systems of Atoms

We shall always assume that $\phi \in C^r(\mathbb{R}^d)$ is a fixed function supported on the reference cell $\overset{\circ}{\theta}$ with $|\overset{\circ}{\theta}| \sim 1$, having some degree of pointwise smoothness $r \in \mathbb{N}$ (in principle, $r = \infty$ is admissible). Moreover, we require that $\phi(x) > 0$ if $x \in \overset{\circ}{\theta}$.

For any $\theta \in \Theta$ we recall (2.7) and set

$$\phi_\theta := \phi \circ A_\theta^{-1}. \tag{3.1}$$

As in PUM we form partitions of unity by defining for any $m \in \mathbb{N}_0$, $\theta \in \Theta_m$

$$\varphi_\theta := \frac{\phi_\theta|_\Omega}{\sum_{\theta' \in \Theta_m} \phi_{\theta'}}, \tag{3.2}$$

where Ω is the domain under consideration. By the properties of ϕ and the cover Θ it follows that

$$0 < c_1 \leq \sum_{\theta \in \Theta_m} \phi_\theta(x) \leq c_2, \quad x \in \Omega, \tag{3.3}$$

where the constants c_1, c_2 depend only on $\mathbf{p}(\Theta)$. Consequently, $\sum_{\theta \in \Theta_m} \varphi_\theta(x) = 1$.

Suppose further that $\{P_\beta : |\beta| = \beta_1 + \cdots + \beta_d \leq k-1\}$ is a basis for Π_k the space of polynomials in d variables of total degree $k-1$, normalized by

$$\|P_\beta \phi\|_{L_\infty(\overset{\circ}{\theta})} = 1. \tag{3.4}$$

Then for $\theta \in \Theta$ we let

$$P_{\theta,\beta} := P_\beta \circ A_\theta^{-1}.$$

Remark 3.1. *As a consequence of the fact that* $|\overset{\circ}{\theta}| \sim 1$ *we have*

$$\|P_\beta \phi\|_{L_p(\overset{\circ}{\theta})} \sim \|P_\beta \phi\|_{L_q(\overset{\circ}{\theta})}, \quad 0 < p, q \leq \infty, \tag{3.5}$$

with constants of equivalence depending only on $p, q, k,$ *and* ϕ.

We define

$$\Phi_m := \{P_{\theta,\beta}\varphi_\theta : \theta \in \Theta_m, |\beta| \leq k-1\} \tag{3.6}$$

and set

$$S_m := \operatorname{span}(\Phi_m) \text{ on } \Omega.$$

Remark 3.2. *It is easy to see that for each $m \in \mathbb{N}_0$*

$$\Pi_k \,|_\Omega \subset S_m,$$

i.e. for every $P \in \Pi_k$ there exists a $g \in S_m$ such that $P|_\Omega = g$.

Our goal is to approximate the solution to (1.1) by linear combinations of the atoms $P_{\theta,\beta}\varphi_\theta$, $\theta \in \Theta$, $|\beta| < k$. This raises a number of well-known practical issues such as the notorious problem of quadrature or the treatment of boundary conditions. In contrast to pure radial basis function approaches the incorporation of essential homogeneous Dirichlet conditions is actually in principle easy and, above all, *local.* In fact, whenever the support of an atom overlaps the boundary one can choose the polynomial factor $P_{\theta,\beta}$ to belong to an ideal whose zero set approximates the corresponding boundary segment. This may even offer better accuracy than common triangular approximations. Since these issues have been addressed elsewhere we concentrate here only on the stability issues related to preconditioning the linear systems resulting from corresponding discretizations.

To this end, it will be important that for each $m \in \mathbb{N}_0$ the collection Φ_m is independent and moreover is stable in L_p. There are several possible ways to go about this. A first one is fomulated as the following property of the atoms:

Property (LLIN): Consider for fixed $0 < \rho_1 < \rho_2$ the collection of all affine maps

$$\mathcal{A}(\rho_1, \rho_2) := \{A : Ax = Mx + b,\ B_{\rho_1}(0) \subset M(B_1(0)) \subset B_{\rho_2}(0)\}.$$

For N_1 (from (C2)) let $\mathcal{A}_0 \subset \mathcal{A}$ be any subset of cardinality $\#\mathcal{A}_0 \leq N_1$ Then for any given pair (ρ_1, ρ_2) as above, any $\mathcal{A}_0$ (of pairwise different affine maps) as above and any ball $B \subset \overset{\circ}{\theta}$ the following *local linear independence property* holds:

$$\sum_{A\in\mathcal{A}_0} \sum_{|\beta|<k} c_{A,\beta}((P_\beta\phi) \circ A)(x) = 0,\ x \in B, \quad \text{implies} \quad c_{A,\beta} = 0,\ A \in \mathcal{A}_0,\ |\beta| < k. \tag{3.7}$$

This means that on any subset of $\overset{\circ}{\theta}$ overlapping affine compositions of ϕ are locally linearly independent.

Scenarios, in which Property (LLIN) can be verified, will be discussed in Section 3.4.

We shall frequently use the obvious fact, that (3.7) is equivalent to the analogous relation for ϕ replaced with φ. Moreover, as a consequence of (C1), we can find a ball $B \subset \overset{\circ}{\theta}$ such that $B_\theta := A_\theta(B)$ satisfies

$$|B_\theta| \geq a_7|\theta|, \tag{3.8}$$

where $0 < a_7 < 1$ also depends only on $\mathbf{p}(\Theta)$.

3.1 Scalewise Stability

In the following we shall briefly write

$$\|g\|_p = \|g\|_{L_p(\Omega)},$$

whenever the domain under consideration is Ω. The first essential building block is the following *levelwise stability* of the partitions of unity.

Theorem 3.3. *Suppose that Property (LLIN) is valid or that the cover Θ is sparse (see Remark 2.2). Then the collection Φ_m ($m \in \mathbb{N}_0$) is linearly independent on Ω and hence forms a basis for $S_m := \mathrm{span}\,(\Phi_m)$. Moreover, any $g \in S_m$ has a unique representation*

$$g = \sum_{\theta \in \Theta_m,\ |\beta|<k} b_{\theta,\beta}(g) P_{\theta,\beta} \varphi_\theta, \tag{3.9}$$

where the dual functionals $b_{\theta,\beta}$ can be defined as follows. For every θ there exists some B_θ with $|B_\theta| \sim |\theta|$ such that

$$b_{\theta,\beta}(f) = \langle f, \tilde{g}_{\theta,\beta} \rangle, \quad \textit{where } \ \mathrm{supp}\,(\tilde{g}_{\theta,\beta}) \subseteq B_\theta, \quad \|\tilde{g}_{\theta,\beta}\|_\infty \le C/|\theta|, \tag{3.10}$$

and C depends only on $\mathbf{p}(\Theta)$. Thus $b_{\theta,\beta}$ is bounded on $L_p(\Omega)$, $1 \le p \le \infty$. As a consequence we have for any $1 \le p \le \infty$

$$|b_{\theta,\beta}(g)| \le c(k, \mathbf{p}(\Theta), p) |\theta|^{-1/p} \|g\|_{L_p(B_\theta)}, \quad \forall \ \ g \in S_m. \tag{3.11}$$

Moreover, for $g \in S_m$, we have

$$\|g\|_p \sim \Big(\sum_{\theta \in \Theta_m,\ |\beta|<k} \|b_{\theta,\beta}(g) P_{\theta,\beta} \varphi_\theta\|_p^p \Big)^{1/p}, \quad 0 < p \le \infty, \tag{3.12}$$

where the constants of equivalence depend only on $k, p, \mathbf{p}(\Theta)$ but not on m and g.

Proof: We shall construct suitable dual functionals by biorthogonalizing local restrictions of interacting atoms. We shall first work under the assumption that property (LLIN) holds. To control the spectrum of the corresponding Gramians we need some preparatory steps. The first one concerns the mutual overlap of atoms from one level. To this end, recall from property (p2) there exists a ball $B_{\bar{\rho}} \subseteq \overset{\circ}{\theta}$ such that

$$|B_{\bar{\rho}}| \ge b_1 |\overset{\circ}{\theta}| \tag{3.13}$$

for some uniform positive constant $b_1 < 1$, where $\bar{\rho}, b_1$ depend only on $\mathbf{p}(\Theta)$. Consider the shrunk versions $B_\ell := B_{(1-\frac{\ell}{2N_1})\bar{\rho}}$ of from (3.13), i.e. $B_0 = B_{\bar{\rho}}$ and $B_{N_1} = B_{\bar{\rho}/2}$. Likewise let $B_{\theta,\ell} := A_\theta(B_\ell)$. Thus, by (p2), we have

$$|B_{\theta,\ell}| \ge b_2 |\theta|, \quad \ell = 0, \dots, N_1, \tag{3.14}$$

for some uniform constant $b_2 > 0$ depending only on $\mathbf{p}(\Theta)$. Furthermore, note that, again by (p2),

$$\theta' \cap B_{\theta,\ell} \ne \emptyset, \quad \theta' \in \Theta_m \implies |\theta' \cap B_{\theta,\ell-1}| \ge b_3 |\theta|, \quad \ell = 1, \dots, N_1, \tag{3.15}$$

where $b_3 > 0$ is another uniform constant depending only on $\mathbf{p}(\Theta)$.

Next observe that there exists an $\ell^* \in \{1, \dots, N_1\}$ such that

$$\begin{aligned} B_{\theta,\ell^*} \cap \theta' &= \emptyset, \ \ \forall \ \ \theta' \in \Theta_m \setminus \{\theta\}, \\ &\text{or} \\ \text{if } \ \theta' \in \Theta_m, \quad \theta' \cap B_{\theta,\ell^*-1} &\ne \emptyset \implies \theta' \cap B_{\theta,\ell^*} \ne \emptyset. \end{aligned} \tag{3.16}$$

In fact, let $\Xi_\ell := \{\theta' \in \Theta_m : \theta' \neq \theta,\ \theta' \cap B_{\theta,\ell} \neq \emptyset\}$. Clearly $\#\Xi_0 \leq N_1$ (see (C2). If Ξ_1 is empty, we set $\ell^* = 1$. If $\#\Xi_1 = \#\Xi_0$ we again set $\ell^* = 1$ and are done. So, it remains to consider the case $\#\Xi_0 > \#\Xi_1 > 0$. Thus, in general, either (3.16) holds for ℓ or $\#\Xi_{\ell+1} < \#\Xi_l$, so that (3.16) holds after at most N_1 steps. We take now ℓ^* as the smallest integer for which (3.16) is valid and set $B := B_{\ell^*-1}$ when the second case in (3.16) holds or $B := B_{\ell^*}$ when the first case is true. Thus, in summary $B_\theta := A_\theta(B)$ for this B satisfies

$$B_\theta \cap \theta' \neq \emptyset, \quad \theta, \theta' \in \Theta_m \quad \Longrightarrow \quad |B_\theta \cap \theta'| \geq b_4|\theta|. \tag{3.17}$$

Now let $\Gamma_\theta := \{\theta' : \theta' \in \Theta_m, \theta' \cap B_\theta \neq \emptyset\}$ and

$$\mathcal{C}_\theta := \{g_{\theta',\beta} := P_{\theta',\beta}\varphi_{\theta'}\chi_{B_\theta} : \theta' \in \Gamma_\theta,\ |\beta| < k\},$$

be the collection of all mth level atoms that overlap B_θ (including those corresponding to θ itself). Note that the $g_{\theta',\beta}$ are defined on all of Ω but vanish outside $B_{\theta'}$. By property (C2), the cardinality of $\mathcal{C}_\theta$ is uniformly bounded by a constant multiple of $N_1 k^d$.

Now consider the local Gramian

$$G_\theta := \left(\langle g_{\theta',\beta}, g_{\theta'',\beta''}\rangle_{B_\theta}\right)_{(\theta',\beta),(\theta'',\beta'')\in\Gamma_\theta},$$

where $\langle v, w\rangle_{B_\theta} := \int_{B_\theta} vw dx$. We shall next show that G_θ is nonsingular and can be used to construct a suitable collection of dual functionals. To this end, note that straightforward substitution yields

$$\langle g_{\theta',\beta}, g_{\theta'',\beta''}\rangle_{B_\theta} = |A_\theta| \int_B \frac{P_\beta(A_{\theta'}^{-1}A_\theta y)\phi(A_{\theta'}^{-1}A_\theta y)P_{\beta''}(A_{\theta''}^{-1}A_\theta y)\phi(A_{\theta''}^{-1}A_\theta y)}{\left(\sum_{\theta'\in\Gamma_\theta}\phi(A_{\theta'}^{-1}A_\theta y)\right)^2} dy. \tag{3.18}$$

Setting $A_\theta y = M_\theta y + x_\theta$, where x_θ is the center of θ and M_θ is the corresponding $(d \times d)$-matrix, one obviously has

$$A_{\theta'}^{-1}A_\theta y = (M_{\theta'}^{-1}M_\theta)y + M_{\theta'}^{-1}(x_\theta - x_{\theta'}).$$

From property (p2) one infers that

$$|M_{\theta'}^{-1}(x_\theta - x_{\theta'})| \leq C, \quad \theta' \in \Gamma_\theta, \tag{3.19}$$

where the constant depends only on $\mathbf{p}(\Theta)$.

Furthermore, considering the singular value decomposition $M_{\theta'}^{-1}M_\theta = U\Sigma V$, U, V orthogonal matrices, the singular values on the diagonal of Σ are contained, on account of property (p2) in a fixed interval $[a_{10}, a_{11}]$ depending only on $\mathbf{p}(\Theta)$ and k, where $a_{10} > 0, a_{11} < \infty$. The orthogonal matrices U, V can also be viewed as elements of a compact finite dimensional manifold. Hence we can write

$$\langle g_{\theta',\beta}, g_{\theta'',\beta''}\rangle_{B_\theta} = |A_\theta| \int_B \frac{P_\beta(A_{\theta',\theta}y)\phi(A_{\theta',\theta}y)P_{\beta''}(A_{\theta'',\theta}y)\phi(A_{\theta'',\theta}y)}{\left(\sum_{\theta'\in\Gamma_\theta}\phi(A_{\theta',\theta}y)\right)^2} dy,$$

where $A_{\theta',\theta} := A_{\theta'}^{-1}A_\theta$, $A_{\theta'',\theta} := A_{\theta''}^{-1}A_\theta$ are affine mappings belonging to $\mathcal{A}(\rho_1, \rho_2)$. Here ρ_1, ρ_2 depend only on $\mathbf{p}(\Theta)$ but not on θ. Moreover the $A_{\theta',\theta}$ are instances of elements in $\mathcal{A}(\rho_1, \rho_2)$

that can be parametrized over some fixed bounded set K of finitely many parameters. On account of (3.15) and (3.17) K is also closed and hence compact. Hence the Gramian can be viewed as a function of the parameters in K which depends only on $\mathbf{p}(\Theta)$. By (3.3) this dependence is continuous. Therefore, each

$$\tilde{G}_\theta := |A_\theta|^{-1} G_\theta \tag{3.20}$$

can be viewed as the value of a continuous matrix valued function at some point in the compact set K. By the above observations, the set $\mathcal{A}_0 := \{A_{\theta',\theta} : \theta' \in \Gamma_\theta\}$ satisfies the requirements in Property (LLIN) for some pair (ρ_1, ρ_2) depending only on $\mathbf{p}(\Theta)$ but not on θ. The determinant of $\tilde{G}_\theta$ is also the evaluation of a continuous function on K. By Property (LLIN) the elements of $\mathcal{C}_\theta$ are linearly independent over B_θ. So the Gramians are always nonsingular and hence their determinants do not vanish in K. Since K is compact they attain their minimum in K that is bounded away from zero from below by some positive constant b_4 depending, in view of (3.17), only on $\mathbf{p}(\Theta)$. Therefore the inverse $\tilde{G}_\theta^{-1}$ exists and is the value of a continuous function on K as well. Let us denote the entries of the inverse $G_\theta^{-1} = |A_\theta|^{-1}\tilde{G}_\theta^{-1}$ by $R_{(\theta',\beta'),(\theta'',\beta'')}$, $(\theta',\beta'), (\theta'',\beta'') \in \Gamma_\theta^k := \Gamma_\theta \times \{\beta \in \mathbb{Z}_+^d : |\beta| < k\}$. Then the functions

$$\tilde{g}_{\theta,\beta} := \sum_{(\theta',\beta')\in\Gamma_\theta^k} R_{(\theta,\beta),(\theta',\beta')} g_{\theta',\beta'} \tag{3.21}$$

which, by construction, are supported on B_θ, form a dual system to Φ_m. In fact,

$$\begin{aligned} \langle \tilde{g}_{\theta,\beta}, P_{\beta^*,\theta^*}\varphi_{\theta^*}\rangle_\Omega &= \langle \tilde{g}_{\theta,\beta}, g_{\beta^*,\theta^*}\rangle_{B_\theta} = \sum_{(\theta',\beta')\in\Gamma_\theta^k} R_{(\theta,\beta),(\theta',\beta')}\langle g_{\theta',\beta'}, g_{\theta^*,\beta^*}\rangle_{B_\theta} \\ &= (G_\theta G_\theta^{-1})_{(\theta,\beta),(\theta^*,\beta^*)} = \delta_{(\theta,\beta),(\theta^*,\beta^*)}, \qquad (\theta,\beta),(\theta^*,\beta^*) \in \Gamma_\theta. \end{aligned} \tag{3.22}$$

To prove that the functionals $b_{\theta,\beta}(g) := \langle \tilde{g}_{\theta,\beta}, g\rangle_{B_\theta}$ satisfy (3.10) it remains to show that

$$\|\tilde{g}_{\theta,\beta}\|_{L_\infty} \le C/|\theta|, \quad \theta \in \Theta_m,\ |\beta| < k, \tag{3.23}$$

where the constant C depends only on the parameters in $\mathbf{p}(\Theta)$. Since by (3.8) the L_∞-norms of the restrictions $g_{\theta',\beta}$ are uniformly bounded from above and away from zero, depending on the parameters in $\mathbf{p}(\Theta)$, (3.23) in turn follows, in view of $\#\Gamma_\theta \le N_1$, $\theta \in \Theta_m$, $m \in \mathbb{N}_0$, as soon as we have shown that

$$|R_{(\theta,\beta),(\theta',\beta')}| \le C/|\theta| \tag{3.24}$$

where again C depends only on $\mathbf{p}(\Theta)$. But this follows, in view of $|A_\theta| \sim |\theta|$ and (3.20), from the fact that $\tilde{G}_\theta^{-1}$ are values of a continuous function whose norm remains bounded on K. This confirms (3.10) under the assumption (LLIN).

When the cover Θ is sparse, the argument is much simpler. In this case we can take B_θ as the largest ball contained in the set Ω_θ which is not intersected by any other $\theta' \in \Theta_m$. By (2.6), we know that $|B_\theta| \sim |\theta|$. Since φ_θ equals one on Ω_θ the local Gramians just involve the polynomials $P_{\theta,\beta}$, $|\beta| < k$. By similar arguments as above these Gramians can be related to a reference domain of unit size to arrive at the same conclusions (3.10). The bound (3.11) follows directly from (3.23). The proof of (3.12) is a standard consequence of (3.22) and (3.11). $\square$

3.2 Quasi-Interpolants

The second crucial ingredient are Quasi-interpolants mapping $L_p(\Omega)$ onto the spaces S_m. We shall distinguish the cases $0 < p < 1$ and $1 \le p \le \infty$, treating the latter case first. Specifically, the mappings

$$Q_m f := \sum_{\theta\in\Theta_m, |\beta|<k} b_{\theta,\beta}(f) P_{\theta,\beta}\varphi_\theta, \quad f \in L_p, \tag{3.25}$$

are, in view of Theorem 3.3, especially (3.11), uniformly bounded projectors from $L_p(\Omega)$ onto S_m for $1 \le p \le \infty$.

Lemma 3.4. *We have*

$$\|Q_m f\|_{L_p(\theta)} \le c_p \|f\|_{L_p(\theta^*)}, \quad \forall \;\; f \in L_p(\Omega),\ 1 \le p \le \infty, \tag{3.26}$$

where for $\theta \in \Theta_m$

$$\theta^* := \bigcup \{\theta' \in \Theta_m : \theta \cap \theta' \ne \emptyset\}.$$

Further immediate consequences of Theorem 3.3 concern the approximation properties of the spaces S_m. To this end, consider the usual forward difference of f in direction h defined by $\Delta_h f(x) := \Delta_h^1 f(x) := f(x+h) - f(x)$ when the line segment $[x, x+h]$ is contained in Ω and by $\Delta_h f(x) = 0$ otherwise. Likewise define for $k > 1$ the kth order forward difference by $\Delta_h^k f(x) := \Delta_h(\Delta_h^{k-1} f(x))$, again provided that $[x, x+kh] \subset \Omega$, while $\Delta_h^k f(x) := 0$ otherwise. Recall that the two versions of the kth L_p-modulus of continuity are then as usual defined as

$$\omega_k(f,\theta)_p := \sup_{t>0} \sup_{|h|\le t} \|\Delta_h^k f\|_{L_p(\theta)}, \qquad \omega_k(f,t)_p := \sup_{|h|\le t} \|\Delta_h^k f\|_{L_p(\Omega)}.$$

Lemma 3.5. *For $f \in L_p(\Omega)$ and $\theta \in \Theta_m$ one has*

$$\|f - Q_m f\|_{L_p(\theta)} \le c \sum_{\theta'\in\Theta_m:\, \theta'\cap\theta\ne\emptyset} \omega_k(f,\theta')_p. \tag{3.27}$$

Hence, one has

$$\|f - Q_m f\|_{L_p(\Omega)} \to 0 \quad \text{as} \;\; m \to \infty. \tag{3.28}$$

Moreover, denoting by $|f|^p_{W^k(L_p(\Omega))} := \sum_{|\beta|=k} \|\partial^\beta f\|^p_{L_p(\Omega)}$ $(p \ge 1)$ the classical kth order Sobolev semi-norm in L_p, an immediate consequence of (3.27) is

$$\|f - Q_m f\|_{L_p(\Omega)} \le c h_m^r |f|_{W^r(L_p(\Omega))}, \quad r \le k, \tag{3.29}$$

where $h_m = \max\{\operatorname{diam}\theta : \theta \in \Theta_m\}$. The constants in (3.27)-(3.29) depend only on $k, \mathbf{p}(\Theta), p$ but not on f, m, θ.

Proof: Estimate (3.27) is an immediate consequence of the locality of the dual functionals, the polynomial reproduction property from Remark 3.2, and a classical Whitney estimate for local polynomial approximation. As for (3.28), it is easy to see (sf. [10]) that

$$\omega_k(f, 2^{-a_0 m/d})_p \sim \Big(\sum_{\theta\in\Theta_m} \omega_k(f,\theta)_p^p\Big)^{1/p}, \tag{3.30}$$

so that (3.28) follows from (3.27) and (3.30). Estimate (3.29) follows from standard estimates for the modulus of continuity given enough smoothness. □

We next introduce a second type of quasi-interpolant, which will be needed when working in L_p with $p < 1$. For $0 < p \le \infty$ and a given cell $\theta \in \Theta$, we let $T_{\theta,p} : L_p(\theta) \to \Pi_{k-1}|_\theta$ be a projector such that

$$\|f - T_{\theta,p} f\|_{L_p(\theta)} \le c\omega_k(f,\theta)_p, \quad f \in L_p(\theta), \tag{3.31}$$

where $c > 0$ depends only on k and the parameters of Θ. Note that $T_{\theta,p} f$ can simply be defined as the best (or near best) approximation to f from Π_{k-1} in $L_p(\theta)$. Then (3.31) is a consequence of Whitney's theorem. Note that we use here that Ω is an extension domain (see Section 1) so that the constant in Whitney's theorem indeed depends only on the shape properties of the θ and thus on $\mathbf{p}(\Theta)$. However, $T_{\theta,p}$ can be realized as a linear projector if $p \ge 1$ by using the Averaged Taylor polynomials, see e.g. [4]. Of course, $T_{\theta,p}$ will be a nonlinear operator if $p < 1$.

We now define the operator $T_{m,p} : L_p(\Omega) \to S_m$ by

$$T_{m,p} f(x) := \sum_{\theta \in \Theta_m} \varphi_\theta(x) T_{\theta,p} f(x), \quad x \in \Omega. \tag{3.32}$$

Evidently, for $0 < p \le \infty$ the operator $T_{m,p} : L_p(\Omega) \to S_m$ is a projector (linear if $p \ge 1$). We next record the most important properties of $T_{\theta,p}$.

Lemma 3.6. *For $f \in L_p(\Omega)$ $(0 < p \le \infty)$ and $\theta \in \Theta_m$ $(m \ge 0)$, we have*

$$\|T_{m,p} f\|_{L_p(\theta)} \le c\|f\|_{L_p(\theta^*)} \tag{3.33}$$

and

$$\|f - T_{m,p} f\|_{L_p(\theta)} \le c \sum_{\theta' \in \Theta_m :\, \theta' \cap \theta \ne \emptyset} \omega_k(f,\theta')_p. \tag{3.34}$$

Moreover,

$$\|f - T_{m,p} f\|_{L_p(\Omega)} \le c\Big(\sum_{\theta \in \Theta_m} \omega_k(f,\theta)_p^p\Big)^{1/p} \le c\omega_k(f, 2^{-a_0 m/d})_p, \tag{3.35}$$

and hence

$$\|f - T_{m,p} f\|_{L_p(\Omega)} \to 0 \quad \text{as} \;\; m \to \infty. \tag{3.36}$$

Here $c > 0$ depends only on p, k, and the parameters of Θ; θ^* is as in Lemma 3.4.

Proof. Estimate (3.33) is an immediate consequence of the definition and the boundedness of $T_{\theta,p}$. Observe that

$$f(x) - T_{m,p} f(x) = \sum_{\theta' \in \Theta_m :\, \theta' \cap \theta \ne \emptyset} \varphi_\theta(x)(f(x) - T_{\theta',p} f(x)), \quad x \in \theta,$$

and hence (3.34) follows by (3.31). Finally (3.35) follows by (3.34) and (3.30). □

Concerning the approximation bounds it would have been enough to work with the $T_{m,p}$ that cover the whole range $0 < p \le \infty$. However, the concrete form of the linear projectors Q_m will be needed in the subsequent section anyway.

3.3 Two-Level Splits

For Schwarz type preconditioners to produce uniformly bounded condition numbers one needs to have stable splittings of the corresponding energy space which, in turn, could be viewed as constructing suitable frames for this space, see e.g. [8, 14]. For such multilevel frames to exist one needs to capture in some sense difference information between successive levels of resolution. In the present framework of MPUHs we cannot expect any nestedness of the spaces S_m. Nevertheless, we shall see in this section that appropriate two-level splits can serve to some extent as substitutes.

To describe such two-level splits let

$$\Lambda_m := \{\lambda = (\eta, \theta, \beta) : \eta \in \Theta_{m+1}, \theta \in \Theta_m, |\theta \cap \eta| \neq 0, |\beta| < k\}, \quad m \geq 0, \tag{3.37}$$

and define

$$F_\lambda := P_{\eta,\beta}\varphi_\eta\varphi_\theta, \qquad \lambda = (\eta, \theta, \beta) \in \Lambda_m. \tag{3.38}$$

Note that

$$\sum_{\eta\in\Theta_{m+1}} \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} \varphi_\eta\varphi_\theta = 1 \quad \text{on } \Omega. \tag{3.39}$$

In order to obtain multilevel decompositions of function spaces based on Θ and the above atoms we shall employ the following two-scale relations of polynomials combined with the partition of unity property of the φ_θ's. To this end, we note that for $\theta \in \Theta_m$, $\eta \in \Theta_{m+1}$

$$P_{\theta,\alpha} = \sum_{|\beta|<k} m^{\theta,\eta}_{\beta,\alpha} P_{\eta,\beta} = \sum_{\eta\in\Theta_{m+1}:\theta\cap\eta\neq\emptyset} \sum_{|\beta|<k} m^{\theta,\eta}_{\beta,\alpha} P_{\eta,\beta}\varphi_\eta, \tag{3.40}$$

where we have used that $\sum_{\eta\in\Theta_{m+1}} \varphi_\eta = 1$.

Theorem 3.7. *For any $f \in L_p(\Omega)$ $(1 \leq p \leq \infty)$ we have (with $Q_{-1} \equiv 0$)*

$$f = \sum_{m=-1}^{\infty} (Q_{m+1}f - Q_m f) = \sum_{m=0}^{\infty} \sum_{\lambda\in\Lambda_m} d_\lambda(f) F_\lambda, \tag{3.41}$$

where for $m^{\theta,\eta}_{\beta,\alpha}$ from (3.40) and the dual functionals $b_{\eta,\beta}(\cdot)$ constructed in Theorem 3.3 one has

$$d_\lambda(f) = b_{\eta,\beta}(f) - \sum_{|\alpha|<k} m^{\theta,\eta}_{\beta,\alpha} b_{\theta,\alpha}(f). \tag{3.42}$$

Proof: The representation (3.41), i.e. the strong convergence of the underlying expansion follows from (3.28). Furthermore, we have

$$\begin{aligned} Q_{m+1}f - Q_m f &= \sum_{\eta\in\Theta_{m+1}} \Big(\sum_{|\beta|<k} b_{\eta,\beta}(f) P_{\eta,\beta}\varphi_\eta\Big) - \sum_{\theta\in\Theta_m} \Big(\sum_{|\alpha|<k} b_{\theta,\alpha}(f) P_{\theta,\alpha}\varphi_\theta\Big) \\ &= \sum_{\theta\in\Theta_m} \varphi_\theta \sum_{\eta\in\Theta_{m+1}} \Big(\sum_{|\beta|<k} b_{\eta,\beta}(f) P_{\eta,\beta}\Big)\varphi_\eta \\ &\quad - \sum_{\theta\in\Theta_m} \Big(\sum_{|\alpha|<k} b_{\theta,\alpha}(f) \sum_{\theta\cap\eta\neq\emptyset} \sum_{|\beta|<k} m^{\theta,\eta}_{\beta,\alpha} P_{\eta,\beta}\varphi_\theta\varphi_\eta\Big) \\ &= \sum_{\eta\in\Theta_{m+1}} \sum_{\theta\in\Theta_m:\ \theta\cap\eta\neq\emptyset} \sum_{|\beta|<k} \Big\{b_{\eta,\beta}(f) - \sum_{|\alpha|<k} m^{\theta,\eta}_{\beta,\alpha} b_{\theta,\alpha}(f)\Big\} P_{\eta,\beta}\varphi_\eta\varphi_\theta, \end{aligned}$$

as claimed. $\square$

For $\lambda = (\eta, \theta, \beta) \in \Lambda_m$ we shall often write $\eta_\lambda = \eta$, $\theta_\lambda = \theta$ and $\beta_\lambda = \beta$.

An important point for later developments is the fact that the representations of the differences $(Q_{m+1} - Q_m)f$ are under certain conditions unique and stable.

Property (LLIN'): Let $\mathcal{A}(\rho_1, \rho_2)$ be defined as in Property (LLIN). In addition we need now a second family of affine maps $\mathcal{A}(\rho_3, \rho_4)$ that are allowed to increase the size of the reference domain by a factor of 2^{a_1} (see (C1)). For subsets B of $\overset{\circ}{\theta}$ as in Property (LLIN), consider any finite subset $\mathcal{R} \subset \mathcal{A}(\rho_1, \rho_2) \times \mathcal{A}(\rho_3, \rho_4)$ of distinct pairs of cardinality at most N_1^2. Then, for any $\mathcal{R}$ with the above property with respect to any list $(\rho_1, \rho_2, \rho_3, \rho_4)$ as above, the two level atoms have the following local linear independence property:

$$\sum_{(A,A') \in \mathcal{R}, |\beta| < k} c_{A,A',\beta} P_\beta(Ay)\phi(Ay)\phi(A'y) = 0, \;\; y \in B, \quad \Longrightarrow \quad c_{A,A',\beta} = 0, \;\; |\beta| < k, (A, A') \in \mathcal{R}.$$

We shall discuss in Section 3.4 relevant situtions where (LLIN') can be verified.

Theorem 3.8. *Suppose that in addition to the assumptions in Theorem 3.3 Property (LLIN') is valid. Then each collection*

$$\{F_\lambda : \lambda \in \Lambda_m\}, \quad m = 0, 1, \ldots,$$

is linearly independent on Ω *and hence forms a basis for*

$$W_m := \mathrm{span}\,\{F_\lambda : \lambda \in \Lambda_m).$$

Moreover, any $g \in W_m$ *has a unique representation*

$$g = \sum_{\lambda \in \Lambda_m} c_\lambda(g) F_\lambda, \tag{3.43}$$

where as in (3.10) the dual functionals c_λ, $\lambda = (\eta, \theta, \beta)$, *have a representation* $c_\lambda(g) = \langle g, c_\lambda \rangle_{B_\eta}$, *for some* $B_\eta \subset \eta$ *which is comparable in size to* η. *Hence the functionals* $c_\lambda(\cdot)$ *are bounded linear functionals on any* $L_p(\Omega)$ *for* $1 \le p \le \infty$ *and satisfy for any* $1 \le p \le \infty$

$$|c_\lambda(g)| \le c(k, p, \mathbf{p}(\Theta)) |\eta|^{-1/p} \|g\|_{L_p(\eta)}, \quad \lambda = (\eta, \theta, \beta), \quad \forall \;\; g \in W_m. \tag{3.44}$$

Proof: Under the given assumptions the construction of the dual functionals is analogous to the one given in the proof of Theorem 3.3. By an analogous reasoning as in the first part of this proof one can establich again the fact that for some constant $b_5 > 0$ and a suitable $B_\eta \subset \eta$ one has

$$B_\eta \cap \eta' \cap \theta \neq \emptyset \quad \Longrightarrow \quad |B_\eta \cap \eta' \cap \theta| \ge b_5 |\eta|. \tag{3.45}$$

Since the remaining assertions are analogous consequences the proof is complete. $\square$

The requirements in (LLIN') can be weakened somewhat when dealing with sparse covers.

Proposition 3.9. *Theorem 3.8 remains valid for a sparse cover if the following is true: For each $\eta \in \Theta_{m+1}$ there exists a neighborhood $\mathcal{N}_\eta \subseteq \Omega_\eta$ such that*

$$\sum_{\substack{\theta \in \Theta_m,\ \theta \cap \mathcal{N}_\eta \neq \emptyset \\ |\beta| < k}} c_{\beta,\theta} P_{\eta,\beta} \phi_\theta(x) = 0, \quad x \in \mathcal{N}_\eta \quad \Longrightarrow \quad c_{\beta,\theta} = 0,\ \theta \in \Theta_m,\ \theta \cap \mathcal{N}_\eta \neq \emptyset,\ |\beta| < k, \tag{3.46}$$

where $\phi_\theta := \phi \circ A_\theta^{-1}$.

Proof: Suppose that, in view of (2.5), (2.6), B_η is again a ball in $\eta \in \Theta_{m+1}$ which is not intersected by any other $\eta' \in \Theta_{m+1}$. Then, since B_η is overlapped only by η itself and since $\varphi_\eta \equiv 1$ on B_η we have

$$\sum_{\lambda' \in \Gamma_\eta^{m,m+1}} c_{\lambda'} F_{\lambda'}(x) = 0 \ \text{ on } B_\eta \quad \Longleftrightarrow \quad \sum_{|\beta'|,k} \sum_{\theta' \cap B_\eta \neq \emptyset} c_{\eta,\beta',\theta'} P_{\eta,\beta'}(x) \varphi_{\theta'}(x) = 0 \ \text{ on } B_\eta.$$

Since the φ_θ and ϕ_θ differ only by one common factor we see that the $F_{\lambda'}$ that overlap B_η are linearly independent on B_η. By the same reasoning as in the proof of Theorem 3.3 we can find a ball $\bar{B}_\eta$ in η whose nonempty intersection with any θ' from Θ_m is substantial, so that the same compactness argument allows us to control the condition of the corresponding local Gramian. $\square$

An immediate consequence of Theorem 3.8 can be stated as follows (see also (3.12)).

Corollary 3.10. *For any $g \in W_m$ we have*

$$\|g\|_p \sim \Big(\sum_{\lambda \in \Lambda_m} \|c_\lambda(g) F_\lambda\|_p^p \Big)^{1/p}, \qquad 0 < p \leq \infty, \tag{3.47}$$

In the following we shall frequently use the following relation

$$\|F_\lambda\|_\tau \sim |\eta_\lambda|^{\frac{1}{\tau} - \frac{1}{p}} \|F_\lambda\|_p, \tag{3.48}$$

which holds for $0 < p, \tau \leq \infty$ with constants depending on p and τ.

3.4 Local Linear Independence

We have already seen in Section 3.1 that property (LLIN) can be weakened somewhat when the cover Θ satisfies in addition to (C1)–(C5) conditions (2.5) and (2.6), i.e. Θ is sparse.

Sparsity is not necessary, as we shall see below, but since it also reduces the computational burden regarding quadrature we shall netxt address this case for two scenarios that might be of practical interest.

Sparsely shifted B-splines: The first scenario is to employ tensor product B-splines of coordinate degree K and maximal smoothness $K-1$ shifted on a regular grid in such a way that polynomial regions match for overlapping supports and that the resulting cover is sparse in the sense of Remark 2.2, see also the example following Remark 2.1. For instance, for cardinal B-splines the supports are shifts of $[0, K+1]^d$ and we could shift on the lattice $L\mathbb{Z}^d$ for some $L \in \mathbb{N}$, $L \le K$ (but close to K to have only a fixed number of overlaps independent of K). Now this situation becomes quite easy to deal with when e.g. $L = K$. To see this, it suffices to consider the coarsest level. Then Ω_θ of each ϕ_θ is a cube of side length $L - 1 = K - 1$. Therefore, when creating higher levels by dyadic subdivisions of the ground lattice, each φ_η for $\eta \in \Theta_{m+1}$ on the next higher level has the property that Ω_η has a nonzero intersection with an Ω_θ for one $\theta \in \Theta_m$. Since on $\mathcal{N}_\eta := \Omega_\eta \cap \Omega_\theta$ the bumps φ_θ and φ_η are constant the validity of (3.46) reduces to the linear independence of the $P_{\eta,\beta}$ which, in view of Proposition 3.9, settles this case completely.

Remark 3.11. *For the above case of sparsely shifted B-splines the assertions of Theorems 3.3 and 3.8 hold.*

Radial local polynomial bumps: To describe a second natural scenario (although less favorable regarding quadrature), suppose that $\overset{\circ}{\theta} = B_1(0)$ is the unit ball and $\phi(x) := ((1 - |x|^2)_+)^K$, where $x_+ := \max\{0, x\}$ and $K \in \mathbb{N}$ is sufficiently large to be specified later. Thus on θ the function φ_θ is a polynomial of degree $2K$.

We shall exploit the fact that the ϕ_θ extend to polynomials $\hat{\phi}_\theta(x) = (1 - |A_\theta^{-1}x|^2)^K$ on all of $\mathbb{R}^d$ and that local linear independence of polynomials is equivalent to their (global) linear independence. Note first that the validity of (3.46) is again immediate if $\mathcal{N}_\eta \cap \Omega_\theta \neq \emptyset$ for some $\theta \in \Theta_m$. In fact, by the sparseness of Θ, no further $\theta' \in \Theta_m$ will then contribute to the linear combination on $\Omega_\theta \cap \mathcal{N}_\eta$. Hence, on $\Omega_\theta \cap \mathcal{N}_\eta$ only the polynomial basis functions interact which again allows us to identify a locally regular Gramian.

The remaining case, namely that for some $\eta \in \Theta_{m+1}$ there exist $\theta \in \Theta_m$ such that

$$\Omega_\eta \cap \Omega_\theta = \emptyset, \quad \Omega_\eta \cap \theta \neq \emptyset, \tag{3.49}$$

would require much more elaboration to rule out a possible linear dependence of overlapping affine compositions of the atoms. At this point it remains an open question whether local linear independence can be guaranteed in this case and a detailed discussion of these issues will be given elsewhere.

Here we are content with sketching a simple way of avoiding this latter difficulty by slightly extending the setting. It will then be relatively easy to ensure the validity of the properties (LLIN) and (LLIN') which is the major motivation for presenting the related arguments in the previous sections. Again we refer to a more detailed exposition in [3]. First this requires, however, expanding slightly the above setting as follows. Instead of taking affine compositions of a *single* ϕ as above, we employ a fixed finite number

$$\phi^j(x) := (1 - |x|^2)_+^{K_j}, \quad j = 1, \ldots, N_2,$$

where the choice of the parameters K_j, N_2 will be explained in a moment. This additional flexibility will allow us though to reduce the requirements (C1) – (C5) significantly and also the sparse covering property is no longer needed. What remains important is that at most a controled number N_1 of atoms overlap at a given point. Then it is possible to color the elements of any two successive levels Θ_m, Θ_{m+1} by at most N_2 colors in such a way that any two θ of the same color are disjoint. Given a fixed numbering of these colors and using a fixed polynomial order k of the polynomial factors P_β, we choose now $K_{j+1} > k+K_j$, $j = 1,\dots,N_2-1$. Thus whenever the supports of a set of atoms have a nonempty intersection, these atoms will have highly differeng polynomial degrees on this intersection. From this it is then easy to see that the atoms are everywhere locally and therefore also globally linearly independent. In principle, a stable collection of dual functionals can then be constructed along similar lines as in Theorems 3.3, 3.8, varying if necessary the supports in the same fashion as in those cases. Since the resulting high polynomial degrees may not favor efficient and accurate calculations we refrein from a detailed discussion at this point but merely use this example to indicate various possible ways of ensuring the above scalewise stability properties.

4 Application to Preconditioning for Elliptic Boundary Value Problems

We now turn to discretizations by means of the above type of partition of unity hierarchies. Thus, for any given $f \in V'$, V a Hilbert space and $a(\cdot,\cdot)$ a symmetric V-elliptic bilinear form (see (1.1)) we consider the problem: Find $u \in V$ such that

$$a(u,v) = \langle f, v\rangle, \qquad \forall \ \ v \in V. \tag{4.1}$$

For simplicity we confine the discussion to the model case $V = H_0^1(\Omega)$. Higher order problems could be treated in an analogous way. The homogeneous boundary conditions are always supposed to be realized in the trial spaces by suitable polynomial factors in the atoms.

Since we shall not deal with discretizations for a fixed level m of resolution but wish to incorporate from the beginning the realization of adaptivity admissible trial functions should in principle, be atoms from *all* levels. More precisely, we shall make use of the atoms F_λ, defined in (3.38) for $\lambda \in \Lambda_m$, $m \in \mathbb{N}_0$. In order to simplify notation we introduce in addition $\Lambda_{-1} := \Theta_0$, set $\Lambda := \bigcup_{m=-1}^{\infty} \Lambda_m$, and use the same notation for the coarse single-level atoms $F_\lambda := P_{\theta,\beta}\varphi_\theta$, $\lambda = (\theta,\beta) \in \Lambda_{-1}$ so that corresponding multilevel expansions take the form $\sum_{\lambda\in\Lambda} a_\lambda F_\lambda$.

We shall place this in the context of *stable splittings* in the theory of *multilevel Schwarz preconditioners* developed by many researchers, see e.g. [14, 12] and the literature cited there. Here we adhere mainly to the findings in [14, 8]. To this end, let $V_\lambda := \operatorname{span}(F_\lambda)$ (see (3.38)) so that $H_0^1(\Omega) := V = \sum_\lambda V_\lambda$. The following is the main result of this section whose proof will be postponed.

Theorem 4.1. *The* $\{V_\lambda\}_{\lambda\in\Lambda}$ *form a* stable splitting *for* V *in the sense that there exist positive finite constants* c_V, C_V*, depending only on* $\mathbf{p}(\Theta)$ *such that*

$$c_V\|v\|_V \le \inf_{v=\sum_\lambda v_\lambda} \left(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-2/d}\|v_\lambda\|_2^2\right)^{1/2} \le C_V\|v\|_V. \tag{4.2}$$

This allows us to invoke the theory of Schwarz methods along the following lines. For $V_0 := S_0 = \operatorname{span}\Phi_0$ define $P_{V_0} : V \to V_0$ and $r_{V_0} \in S_0$ by

$$a(P_{V_0}v, F_\lambda) = a(v, F_\lambda), \quad (r_{V_0}, F_\lambda)_{L_2} = \langle f, F_\lambda\rangle, \quad \lambda \in \Lambda_0 = \Theta_0.$$

Moreover, introducing the auxiliary bilinear forms:

$$b_\lambda(v,w) := |\eta_\lambda|^{-2/d}(v,w)_{L_2}, \quad v,w \in V_\lambda, \quad \lambda \in \Lambda \setminus \Lambda_0, \tag{4.3}$$

we endow the spaces V_λ with the norms $\|v\|_{V_\lambda} := (b_\lambda(v,v))^{1/2}$ and define the linear operator $P_{V_\lambda} : V \to V_\lambda$ and $f_\lambda \in V_\lambda$ by

$$\begin{array}{rcl} |\eta_\lambda|^{-2/d}(P_{V_\lambda}v, F_\lambda)_{L_2} &=& a(v, F_\lambda), \\ |\eta_\lambda|^{-2/d}(f_\lambda, F_\lambda)_{L_2} &=& \langle f, F_\lambda\rangle. \end{array} \tag{4.4}$$

Thus, as usual,

$$P_{V_\lambda}v = a_\lambda(v)F_\lambda, \qquad f_\lambda = r_\lambda(f)F_\lambda, \tag{4.5}$$

with

$$a_\lambda(v) = \frac{|\eta_\lambda|^{2/d}a(v,F_\lambda)}{\langle F_\lambda, F_\lambda\rangle}, \qquad r_\lambda(f) = \frac{|\eta_\lambda|^{2/d}\langle f, F_\lambda\rangle}{\langle F_\lambda, F_\lambda\rangle}. \tag{4.6}$$

The following statements are now an immediate consequence of the results in [8, 14].

Theorem 4.2. *Problem (4.1) is equivalent to the operator equation*

$$P_V u = \bar{f}, \tag{4.7}$$

where

$$P_V := P_{V_0} + \sum_{\lambda\in\Lambda\setminus\Lambda_0} P_{V_\lambda}, \qquad \bar{f} := r_{V_0} + \sum_{\lambda\in\Lambda\setminus\Lambda_0} f_\lambda. \tag{4.8}$$

Moreover, the spectral condition number $\kappa(P_V)$ of the additive Schwarz operator P_V satisfies

$$\kappa(P_V) \le \frac{C_a C_V}{c_a c_V}, \tag{4.9}$$

where c_A, C_A, c_V, C_V are the constants from (1.1) and (4.2).

This latter fact implies that simple iterative schemes, such as Richardson iterations,

$$u^{n+1} = u^n + \alpha(\bar{f} - P_V u^n), \quad n = 0,1,2,\ldots, \tag{4.10}$$

converge with a fixed error reduction rate per step. More specifically, suppose that $u^n = \sum_{\lambda\in\Lambda} u^n_\lambda F_\lambda$ with coefficient array $\mathbf{u}^n = (u^n_\lambda)_{\lambda\in\Lambda}$, (4.10) can be rephrased, in view of (4.5), (4.6) as

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \alpha(\mathbf{r} - \mathbb{A}\mathbf{u}^n), \quad \mathbb{A}_{\lambda,\lambda'} = |\eta_\lambda|^{2/d}\|F_\lambda\|_2^{-2} a(F_\lambda, F_{\lambda'}), \quad \lambda,\lambda' \in \Lambda\setminus\Lambda_0. \tag{4.11}$$

A few comments are in order. First of all, the above operator equation (4.7) is formulated in the full infinite dimensional space. Alternatively, restricting the summation to an a priori chosen finite subset $\bar\Lambda$ of Λ e.g. $\bar\Lambda := \bigcup_{m\le M}\Lambda_m$ we obtain a finite dimensional discrete problem whose condition obviously fulfills the same bound, uniformly in the size and choice of $\bar\Lambda$. In this sense we have an asymptotically optimal preconditioner.

On the other hand, it is conceptually useful to consider the full infinite dimensional problem (4.7). In this case (4.10) is to be understood as an *idealized* scheme whose numerical implementation requires appropriate *approximate* applications of the (infinite dimensional) operator P_V quite in the spirit of [2]. This can be done by computing in addition to solving the coarse scale problem on $S_0 = V_0$ only finitely many but properly selected components P_{V_λ} each requiring only the solution of a one-dimensional problem. This hints at the adaptive potential of such an approach similar to the developments in [2]. Roughly speaking, one could try to monitor the size of the components of the weighted residual $\alpha(\bar{f} - P_V u^n)$ so as to replace it within a suitable tolerance by a vector of possibly small support. Thereby one would try to keep the supports of the approximations $\mathbf{u}^n$ as small as possible again within a desired gain of accuracy. This, in turn, raises the question which accuracy can be achieved at best when using linear combinations of at most N of the atoms, i.e. we are interested in the error of *best N-term approximation*

$$\sigma_{N,X}(v) := \inf \{\|v - \sum_{\lambda \in \tilde{\Lambda}} a_\lambda F_\lambda\|_X : a_\lambda \in \mathbb{R},\ \#\tilde{\Lambda} \leq N\}. \tag{4.12}$$

To see whether any adaptive strategy could offer a gain over simple uniform refinements it is interesting to understand the corresponding *approximation spaces*

$$\mathcal{A}_X^s := \{v \in V : |v|_{\mathcal{A}_X^s} := \sup_{N \in \mathbb{N}} N^s \sigma_{N,X}(v) < \infty\}. \tag{4.13}$$

A more thorough discussion of related adaptive solution schemes will be given elsewhere. The remainder of this note is devoted to the proof of the above stable splittings and to a short discussion of best N-term approximation in the present context.

5 Smoothness Spaces and Best N-Term Approximation

5.1 B-spaces and Besov spaces

For variational problems of the type considered in the previous section the energy space V is typically a Sobolev space. A common strategy for establishing the stability (4.2) of the splitting $\{F_\lambda\}_{\lambda \in \Lambda}$ required in Theorem 4.2 in this context is to exploit that the Sobolev spaces $H^t(\Omega)$ (or corresponding subspaces with vanishing traces) agree with the Besov spaces $B_2^t(L_2(\Omega))$ with equivalent norms and that the Besov norms are more suitable for analyzing multilevel splittings. Moreover, Besov spaces on $L_p(\Omega)$ for $p \neq 2$ are relevant for the analysis of nonlinear approximation such as best N-term approximation. Let us briefly recall that the Besov space $B_q^\alpha(L_p(\Omega))$, with $\alpha > 0$ and $0 < p, q \leq \infty$, is usually defined as the set of all functions $f \in L_p(\Omega)$ such that

$$|f|_{B_q^\alpha(L_p(\Omega))} := \Big(\int_0^\infty (t^{-\alpha} \omega_k(f,t)_p)^q \frac{dt}{t} \Big)^{1/q} < \infty \tag{5.1}$$

with the usual modification when $q = \infty$. As before $\omega_k(f,t)_p$ is the kth modulus of smoothness of f in L_p over Ω. The norm in $B_q^\alpha(L_p(\Omega))$ is defined by

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := |\Omega|^{-\alpha/d} \|f\|_{L_p} + |f|_{B_q^\alpha(L_p(\Omega))}.$$

It is not hard to see that

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \Big(\sum_{j=0}^\infty \Big(2^{\alpha j}\omega_k(f,2^{-j})_p\Big)^{q/p}\Big)^{1/q} \tag{5.2}$$

Moreover, following [10], the moduli of smoothness can be localized which allows us to to related the Besov norms to the cover Θ from Section 2 by verifying that

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \Big(\sum_{m=0}^\infty \Big(\sum_{\theta\in\Theta_m} |\theta|^{-\alpha p/d}\omega_k(f,\theta)_p^p\Big)^{q/p}\Big)^{1/q}. \tag{5.3}$$

To see how this, in turn, can be related to norms of the type appearing in (4.2), it will be convenient to introduce next a scale of "smoothness spaces" (B-spaces) induced by multilevel covers Θ as described in Section 2. The construction of these spaces is inspired by previous work referring to a different setting, see [4, 10, 13]. As before we assume that Ω is a bounded extension domain in $\mathbb{R}^d$ as explained in Section 1.

As for Besov spaces we could incorporate a third fine tuning parameter. Since this will not be needed in the present applications we shall be content with the following technically simpler version.

The following first version defines the B-space $\mathcal{B}_p^s(\Theta)$ via atomic decompositions which will provide our link to the stable splittings in Theorem 4.1. More precisely, the B-space $\mathcal{B}_p^s(\Theta)$, $s>0$, $0<p\le\infty$, is defined as the set of all functions $f\in L_p(\Omega)$ such that

$$\|f\|_{\mathcal{B}_p^s(\Theta)} := \inf_{f=\sum_{\lambda\in\Lambda} a_\lambda F_\lambda} \Big(\sum_{\lambda\in\Lambda} |\theta_\lambda|^{-sp}\|a_\lambda F_\lambda\|_p^p\Big)^{1/p}, \tag{5.4}$$

where the infimum is taken over all representations $f=\sum_{\lambda\in\Lambda} a_\lambda F_\lambda$ in $L_p(\Omega)$. Here $\Lambda := \cup_{m=0}^\infty \Lambda_m$, $\Lambda_0=\Theta_0$.

A second approach to the B-spaces $\mathcal{B}_p^s(\Theta)$, that will help us to interrelate the above norms, is through quasi-interpolants. For $f\in L_p(\Omega)$, $1\le p\le\infty$, we have by Theorem 3.7

$$f = Q_0 f + \sum_{m=0}^\infty (Q_{m+1}f - Q_m f) = \sum_{m=0}^\infty \sum_{\lambda\in\Lambda_m} d_\lambda(f)F_\lambda. \tag{5.5}$$

Whenever $0<p<1$, however, we need employ the quasi-interpolats $T_{m,p}$ from (3.32). It follows by Lemma 3.6 that for $f\in L_p(\Omega)$, $0<p\le\infty$,

$$f = T_{0,p} f + \sum_{m=0}^\infty (T_{m+1,p}f - T_{m,p} f) = \sum_{m=0}^\infty \sum_{\lambda\in\Lambda_m} d_\lambda(f)F_\lambda \quad \text{in} \quad L_p. \tag{5.6}$$

Here we denoted again by $\{d_\lambda(f)\}_{\lambda\in\Lambda_m}$ the coefficients in the representation of $T_{m+1,p}f - T_{m,p}f$ in W_m. We define

$$\|f\|_{\mathcal{B}_p^s(\Theta)}^Q := \Big(\sum_{\lambda\in\Lambda} |\theta_\lambda|^{-sp}\|d_\lambda(f)F_\lambda\|_p^p\Big)^{1/p}, \tag{5.7}$$

where $\{d_\lambda(f)\}_{\lambda\in\Lambda}$ come from (5.5) if $1 \le p \le \infty$ and from (5.6) if $0 < p < 1$. Notice that the coefficients from (5.6) could be used in both cases.

These B-spaces are conveniently linked to Besov spaces by introducing the third version through the semi-norm

$$|f|^{\omega}_{\mathcal{B}^s_p(\Theta)} := \Big(\sum_{\theta\in\Theta} |\theta|^{-ps}\omega_k(f,\theta)_p^p\Big)^{1/p} < \infty, \tag{5.8}$$

where $\omega_k(f,\theta)_p$ is again the kth modulus of smoothness of f on θ in L_p. We set

$$\|f\|^{\omega}_{\mathcal{B}^s_p(\Theta)} := |\Omega|^{-s}\|f\|_p + |f|^{\omega}_{\mathcal{B}^s_p(\Theta)}. \tag{5.9}$$

Evidently, $\|\cdot\|^{\omega}_{\mathcal{B}^s_p(\Theta)}$ is a norm if $p \ge 1$ and quasi-norm otherwise. This norm now depends on one more parameter $k \ge 1$ which we shall not indicate explicitly in the notation before we clearly exhibit its role. We shall assume at this point, however, that $k \le r$, where $r > 0$ is the smoothness of our building blocks ϕ (see the beginning of Section 3).

A glance at (5.3) reveals that the latter norm is just the Besov norm where the smoothness index is rescaled, i.e. s plays the role of α/d.

Remark 5.1. *For $0 < s < k/d$, we have $\mathcal{B}^s_p(\Theta) = B^{ds}_p(L_p(\Omega))$ and for f in this space one has*

$$\|f\|_{\mathcal{B}^s_p(\Theta)} \sim \|f\|_{B^{ds}_p(L_p(\Omega))}.$$

Without further mentioning we assume in the following that Property (LLIN') or the hypotheses of Proposition 3.9 are valid.

The main result of this section concerns the following interrelation of the above norms.

Theorem 5.2. *Let $s > 0$, $0 < p \le \infty$, and $k \ge 1$.*

(a) If $f \in \mathcal{B}^s_p(\Theta)$, then

$$\|f\|_{\mathcal{B}^s_p(\Theta)} \le \|f\|^{Q}_{\mathcal{B}^s_p(\Theta)} \lesssim \|f\|^{\omega}_{\mathcal{B}^s_p(\Theta)} \tag{5.10}$$

(b) The norms $\|\cdot\|_{\mathcal{B}^s_p(\Theta)}$, $\|\cdot\|^{Q}_{\mathcal{B}^s_p(\Theta)}$, and $\|\cdot\|^{\omega}_{\mathcal{B}^s_p(\Theta)}$, defined in (5.4),(5.7) and (5.9), are equivalent for $0 < s < k/d$. Here the constants depend only on s, p, k, and the parameters in $\mathbf{p}(\Theta)$ of Θ.

Proof: As for (a), in view of the special decomposition $f = \sum_m (Q_m - Q_{m-1})f$, the first inequality is trivial. To confirm the second inequality, we recall that, by (3.47)

$$\begin{aligned}\sum_{\lambda\in\Lambda_m} \|d_\lambda(f)F_\lambda\|_p^p \quad &\sim \quad \|(Q_{m+1}-Q_m)f\|_p^p \le \sum_{\theta\in\Theta_m} \|(Q_{m+1}-Q_m)f\|^p_{L_p(\theta)} \\ &\lesssim \quad \sum_{\theta\in\Theta_m} \omega_k(f,\theta)_p^p,\end{aligned}$$

where we have used in the last step (3.27), (C2), (C3), (C5) as well as standard properties of the modulus of continuity. The right-hand-side inequality in (5.10) is now an immediate consequence of definition (5.8).

To confirm (b) it remains to show that

$$\|f\|^{\omega}_{\mathcal{B}^s_p(\Theta)} \lesssim \|f\|_{\mathcal{B}^s_p(\Theta)}. \tag{5.11}$$

Consider first the easier case $p \le 1$. For any decomposition $f = \sum_{\lambda\in\Lambda} a_\lambda F_\lambda$ in L_p and $\theta \in \Theta$, we have

$$\begin{aligned}\omega_k(f,\theta)_p^p &\lesssim \omega_k\Big(\sum_{|\eta_\lambda|>|\theta|} a_\lambda F_\lambda, \theta\Big)_p^p + \Big\|\sum_{|\eta_\lambda|\le|\theta|} a_\lambda F_\lambda\Big\|^p_{L_p(\theta^*)} \\ &\le \sum_{|\eta_\lambda|>|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset} |a_\lambda|^p \omega_k(F_\lambda,\theta)_p^p + \sum_{|\eta_\lambda|\le|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset} \|a_\lambda F_\lambda\|_p^p. \end{aligned} \tag{5.12}$$

Estimating $\omega_k(f,\theta)_p^p$ requires the following simple technical observations. Recalling that by the properties (C1)–(C5) our normalization ensures that $\|F_\lambda\|_\infty \sim 1$, one derives that

$$\Big\|\partial^\alpha F_\lambda\Big\|_\infty \lesssim |\eta_\lambda|^{-|\alpha|/d}, \quad |\alpha| \le k.$$

Hence for any $h \in \mathbb{R}^d$, $|h| \le \operatorname{diam}\theta \sim |\theta|^{1/d}$ (see p1, p2, (C1))

$$\begin{aligned}\omega_k(F_\lambda,\theta)_p^p &\lesssim |h|^{kp}\Big\|\Big(\frac{\partial}{\partial h}\Big)^k F_\lambda\Big\|^p_\infty |\theta| \le |\theta|^{kp/d}|\eta_\lambda|^{-pk/d}|\theta| \\ &= \Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{kp/d}|\theta| \lesssim \Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\frac{kp}{d}+1}\|F_\lambda\|_p^p, \end{aligned} \tag{5.13}$$

where we used that $\|F_\lambda\|_p \sim |\eta_\lambda|^{1/p}$ (see (3.48)) due to the normalization $\|F_\lambda\|_\infty \sim 1$.

Therefore, one has by (5.8) and (5.12)-(5.13),

$$\begin{aligned}(|f|^{\omega}_{\mathcal{B}^s_p(\Theta)})^p &= \sum_{\theta\in\Theta} |\theta|^{-sp}\omega_k(f,\theta)_p^p \\ &\lesssim \sum_{\theta\in\Theta}|\theta|^{-sp}\sum_{|\eta_\lambda|>|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\frac{kp}{d}+1}\|a_\lambda F_\lambda\|_p^p \\ &\quad + \sum_{\theta\in\Theta}|\theta|^{-sp}\sum_{|\eta_\lambda|\le|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset}\|a_\lambda F_\lambda\|_p^p \\ &=: \Sigma_1 + \Sigma_2. \end{aligned} \tag{5.14}$$

Furthermore, we have by a geometric series argument and (C2), (C5), that

$$\Sigma_2 \le \sum_{\lambda\in\Lambda}\|a_\lambda F_\lambda\|_p^p \sum_{|\theta|\ge|\eta_\lambda|,\,\theta\cap\eta_\lambda\neq\emptyset}|\theta|^{-sp} \lesssim \sum_{\lambda\in\Lambda}\|a_\lambda F_\lambda\|_p^p|\eta_\lambda|^{-sp}. \tag{5.15}$$

As for the first part, we have

$$\Sigma_1 \le \sum_{\lambda\in\Lambda}\|a_\lambda F_\lambda\|_p^p \underbrace{\sum_{|\theta|<|\eta_\lambda|,\,\theta\cap\eta_\lambda\neq\emptyset}|\theta|^{-sp}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\frac{kp}{d}+1}}_{:=w_\lambda}. \tag{5.16}$$

Now we invoke properties (C1) and (C5) of the cover Θ which ensures that for $\eta_\lambda \in \Theta_m$ and $\theta \in \Theta_{m+l}$, $l \geq 1$, one has $|\theta|/|\eta_\lambda| \lesssim 2^{-la_0}$. Moreover, the number of cells $\theta \in \Theta_{m+l}$ whose support intersect η_λ is bounded by a constant multiple of 2^{la_0}. Hence, one obtains

$$\begin{aligned} w_\lambda &= |\eta_\lambda|^{-sp} \sum_{|\theta|<|\eta_\lambda|,\, \theta\cap\eta_\lambda\neq\emptyset} |\theta|^{-sp} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{kp}{d}+1-sp} \sim |\eta_\lambda|^{-sp} \sum_{l=1}^{\infty} 2^{la_0} 2^{-la_0\left(\frac{kp}{d}+1-sp\right)} \\ &= |\eta_\lambda|^{-sp} \sum_{l=1}^{\infty} 2^{-la_0\left(\frac{k}{d}-s\right)p} \sim |\eta_\lambda|^{-sp}, \end{aligned} \tag{5.17}$$

using that $s < k/d$. Inserting this into (5.16) and combining (5.16) with (5.15), implies $|f|^{\omega}_{\mathcal{B}^s_p(\Theta)} \lesssim \|f\|_{\mathcal{B}^s_p(\Theta)}$ for $s < k/d$ and $p \leq 1$.

The estimate $|\Omega|^{-s}|f|_p \lesssim \|f\|_{\mathcal{B}^s_p(\Theta)}$ follows similarly as the estimate of Σ_2 above but is easier and its proof will be omitted. This completes the proof of (5.11) in the case $p \leq 1$.

We next prove (5.11) in the case $p > 1$. Consider any decomposition $f = \sum_{\lambda\in\Lambda} a_\lambda F_\lambda$ in L_p. Noticing that (5.13) holds for $0 < p \leq \infty$, we have for $\theta \in \Theta$

$$\begin{aligned} \omega_k(f,\theta)_p &\lesssim \omega_k\Big(\sum_{|\eta_\lambda|>|\theta|} a_\lambda F_\lambda, \theta\Big)_p + \Big\|\sum_{|\eta_\lambda|\leq|\theta|} a_\lambda F_\lambda\Big\|_{L_p(\theta^*)} \\ &\leq \sum_{|\eta_\lambda|>|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} |a_\lambda|\omega_k(F_\lambda,\theta)_p + \Big\|\sum_{|\eta_\lambda|\leq|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda\Big\| \\ &\lesssim \sum_{|\eta_\lambda|>|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{k}{d}+\frac{1}{p}} \|a_\lambda F_\lambda\|_p + \Big\|\sum_{|\eta_\lambda|\leq|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda\Big\|. \end{aligned} \tag{5.18}$$

Now by (5.8) and (5.18), we infer

$$\begin{aligned} (|f|^{\omega}_{\mathcal{B}^s_p(\Theta)})^p &= \sum_{\theta\in\Theta} |\theta|^{-sp}\omega_k(f,\theta)_p^p \\ &\lesssim \sum_{\theta\in\Theta} |\theta|^{-sp}\Big[\sum_{|\eta_\lambda|>|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{k}{d}+\frac{1}{p}} \|a_\lambda F_\lambda\|_p\Big]^p \\ &\quad + \sum_{\theta\in\Theta} |\theta|^{-sp}\Big\|\sum_{|\eta_\lambda|\leq|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda\Big\|_p^p \\ &=: \Sigma_1 + \Sigma_2. \end{aligned} \tag{5.19}$$

For the first sum, we have

$$\begin{aligned} \Sigma_1 &= \sum_{\theta\in\Theta}\Big[\sum_{|\eta_\lambda|>|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{k}{d}-s+\frac{1}{p}} |\eta_\lambda|^{-s}\|a_\lambda F_\lambda\|_p\Big]^p \\ &= \sum_{\theta\in\Theta}\Big[\sum_{|\eta_\lambda|>|\theta|,\, \eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta+\frac{1}{p}} A_\lambda\Big]^p, \end{aligned} \tag{5.20}$$

where $2\delta := k/d - s > 0$ and $A_\lambda := |\eta_\lambda|^{-s}\|a_\lambda F_\lambda\|_p$. Applying Hölder's inequality, we get

$$\Sigma_1 \;\le\; \sum_{\theta\in\Theta}\Big[\sum_{|\eta_\lambda|>|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\delta p'}\Big]^{p/p'}\sum_{|\eta_\lambda|>|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\delta p+1}A_\lambda^p, \tag{5.21}$$

where $1/p+1/p' = 1$. Similarly as above for $\theta \in \Theta_m$ and $\eta_\lambda \in \Theta_{m-\nu}$ one has $|\theta|/|\eta_\lambda| \lesssim 2^{-\nu a_0}$. Consequently,

$$\sum_{|\eta_\lambda|>|\theta|,\,\eta_\lambda\cap\theta\neq\emptyset}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\delta p'} \;\lesssim\; \sum_{\nu=0}^{\infty}2^{-\nu a_0\delta p'} \;\lesssim\; 1. \tag{5.22}$$

We use this in (5.21) and switch the order of summation to obtain

$$\Sigma_1 \;\lesssim\; \sum_{\lambda\in\Lambda}A_\lambda^p\sum_{|\theta|<|\eta_\lambda|,\,\theta\cap\eta_\lambda\neq\emptyset}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\delta p+1}. \tag{5.23}$$

Fix $\lambda \in \Lambda$ and assume that $\eta_\lambda \in \Theta_j$. Exactly as in (5.17) we use that the number of cells $\theta \in \Theta_{j+l}$ whose support intersect η_λ is bounded by $c2^{la_0}$ to obtain

$$\sum_{|\theta|<|\eta_\lambda|,\,\theta\cap\eta_\lambda\neq\emptyset}\Big(\frac{|\theta|}{|\eta_\lambda|}\Big)^{\delta p+1} \;\lesssim\; \sum_{l=0}^{\infty}\sum_{\theta\in\Theta_{j+l},\,\theta\cap\eta_\lambda\neq\emptyset}2^{-la_0(1+\delta p)} \;\lesssim\; \sum_{l=0}^{\infty}2^{-la_0\delta p} \;\lesssim\; 1.$$

Inserting this in (5.23) we get

$$\Sigma_1^{1/p} \;\lesssim\; \|f\|_{\mathcal{B}^s_p(\Theta)}. \tag{5.24}$$

We now estimate Σ_2. Note first that by (C2) it follows that

$$\Big\|\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset}a_\lambda F_\lambda\Big\|_p^p \;\lesssim\; \sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset}\|a_\lambda F_\lambda\|_p^p, \quad \text{if} \;\; \theta\in\Theta_m, \quad \nu\ge 0.$$

Hence

$$\begin{aligned}\Sigma_2 \;&\lesssim\; \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m}|\theta|^{-sp}\Big[\sum_{\nu=0}^{\infty}\Big(\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset}\|a_\lambda F_\lambda\|_p^p\Big)^{1/p}\Big]^p\\ &=\; \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m}\Big[\sum_{\nu=0}^{\infty}\Big(\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset}\Big(\frac{|\eta_\lambda|}{|\theta|}\Big)^{sp}|\eta_\lambda|^{-sp}\|a_\lambda F_\lambda\|_p^p\Big)^{1/p}\Big]^p.\end{aligned}$$

As above we denote $A_\lambda := |\eta_\lambda|^{-s}\|a_\lambda F_\lambda\|_p$ and use that $|\eta_\lambda|/|\theta| \lesssim 2^{-\nu a_0}$ if $\theta \in \Theta_m$, $\eta_\lambda \in \Theta_{m+\nu}$ to obtain

$$\begin{aligned}\Sigma_2 \;&\lesssim\; \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m}\Big[\sum_{\nu=0}^{\infty}2^{-\nu a_0 s/2}\Big(\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset}\Big(\frac{|\eta_\lambda|}{|\theta|}\Big)^{sp/2}A_\lambda^p\Big)^{1/p}\Big]^p\\ &=:\; \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m}\sigma_\theta.\end{aligned} \tag{5.25}$$

Now applying Hölder's inequality we have

$$\begin{aligned}\sigma_\theta &\le \Big(\sum_{\nu=0}^{\infty} 2^{-\nu a_0 sp'/2}\Big)^{p/p'} \sum_{\nu=0}^{\infty} \sum_{\eta_\lambda\in\Theta_{m+\nu},\, \eta_\lambda\cap\theta\neq\emptyset} \Big(\frac{|\eta_\lambda|}{|\theta|}\Big)^{sp/2} A_\lambda^p \\ &\lesssim \sum_{\nu=0}^{\infty} \sum_{\eta_\lambda\in\Theta_{m+\nu},\, \eta_\lambda\cap\theta\neq\emptyset} \Big(\frac{|\eta_\lambda|}{|\theta|}\Big)^{sp/2} A_\lambda^p.\end{aligned}$$

Substituting this in (5.25) and switching the order of summation, we obtain

$$\Sigma_2 \quad\lesssim\quad \sum_{\lambda\in\Lambda} A_\lambda^p \sum_{|\theta|\ge|\eta_\lambda|,\, \theta\cap\eta_\lambda\neq\emptyset} \Big(\frac{|\eta_\lambda|}{|\theta|}\Big)^{sp/2}.$$

Exactly as in (5.22) the second sum above can be bounded from above by a constant, which implies $\Sigma_2^{1/p} \le \|f\|_{\mathcal{B}_p^s(\Theta)}$. This coupled with (5.24) yields $|f|^{\omega}_{\mathcal{B}_p^s(\Theta)} \lesssim \|f\|_{\mathcal{B}_p^s(\Theta)}$.

The estimate $|\Omega|^{-s}|f|_p \lesssim \|f\|_{\mathcal{B}_p^s(\Theta)}$ is similar to the estimate of Σ_2 above but is easier and its proof will be omitted. The proof of (5.11) is complete. □

An immediate further consequence of (5.3) is the following fact which, in particular, completes the proof of Theorem 4.1.

Corollary 5.3. *Under the above assumptions the norms* $\|\cdot\|_{B_p^{sd}(L_p(\Omega))}$, $\|\cdot\|^{\omega}_{\mathcal{B}_p^s(\Theta)}$, $\|\cdot\|_{\mathcal{B}_p^s(\Theta)}$, *and* $\|\cdot\|^{Q}_{\mathcal{B}_p^s(\Theta)}$, *defined in* (5.1), (5.9), (5.4) *and* (5.7), *are equivalent for* $0 < s < k/d$ *and admissible* s, p. *Since the norms* $a(\cdot,\cdot)^{1/2}$ *and* $\|\cdot\|_{H^1(\Omega)}$ *are equivalent, employing the well known fact that*

$$\|\cdot\|_{H^1(\Omega)} \sim \|\cdot\|_{B_2^1(L_2(\Omega))},$$

Theorem 4.1 follows.

5.2 Best N-Term Approximation

In this section we collect some consequences of the above findings regarding *best N-term approximation* based on the system $\{F_\lambda : \lambda\in\Lambda\}$, cf. (4.12). In particular, this would clarify what could be achieved at best by an adaptive strategy based on (4.10). For approximation in $X = L_p(\Omega)$, $0 < p < \infty$, we can resort to the general results in [10]. In fact, conditions (i), (ii) in [10, Theorem 3.3] are readily seen to be satisfied by the terms $d_\lambda F_\lambda$. Then, setting

$$\alpha := \frac{1}{\tau} - \frac{1}{p}, \tag{5.26}$$

we have for $f = \sum_{\lambda\in\Lambda} d_\lambda(f) F_\lambda$

$$\begin{aligned}\|f\|^{Q}_{\mathcal{B}_\tau^\alpha(\Theta)} &= \Big(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-\alpha\tau} \|d_\lambda(f)F_\lambda\|_\tau^\tau\Big)^{1/\tau} \sim \Big(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-\alpha\tau} |\eta_\lambda|^{1-\tau/p} \|d_\lambda(f)F_\lambda\|_p^\tau\Big)^{1/\tau} \\ &= \Big(\sum_{\lambda\in\Lambda} \|d_\lambda(f)F_\lambda\|_p^\tau\Big)^{1/\tau},\end{aligned} \tag{5.27}$$

where we have used (5.26) and (3.48). Thus, whenever $f \in \mathcal{B}_\tau^\alpha(\Theta)$ for any $\alpha > 0$ and α, τ related through (5.26), assumption (3.6) in [10, Theorem 3.3] is satisfied. Therefore, [10, Theorem 3.4] ensures that for the N largest terms $\|d_{\lambda_1} F_{\lambda_1}\|_p \geq \|d_{\lambda_2} F_{\lambda_2}\|_p \geq \cdots \geq \|d_{\lambda_N} F_{\lambda_N}\|_p$ and $S_N := \sum_{j=1}^N d_{\lambda_j} F_{\lambda_j}$ we have

$$\|f - S_N\|_p \lesssim N^{-\alpha} \|f\|_{\mathcal{B}_\tau^\alpha(\Theta)}.$$

Denoting by $\sigma_{N,L_p}(f)$ the best N-term nonlinear approximation from $\{F_\lambda\}_{\lambda \in \Lambda}$ in $L_p(\Omega)$, we obtained the Jackson estimate

$$\sigma_{N,L_p}(f) \lesssim N^{-\alpha} \|f\|_{\mathcal{B}_\tau^\alpha(\Theta)}, \quad N \in \mathbb{N}, \quad f \in \mathcal{B}_\tau^\alpha(\Theta). \tag{5.28}$$

In particular, when $\alpha < k/d$, the regularity assumption $f \in B_\tau^{\alpha d}(L_\tau(\Omega))$ ensures the rate $\sigma_{N,L_p}(f) \lesssim N^{-\alpha}$. Note that (5.26) means that $B_\tau^{\alpha d}(L_\tau(\Omega))$ is in some sense the largest space of smoothness αd that is still embedded in L_p. We do not address here corresponding inverse estimates which are much more involved.

In the context of Section 4 it is perhaps more interesting to understand best N-term approximation in $X = H_0^1(\Omega)$, the energy space of second order elliptic problems, which is in some sense an easier problem.

Theorem 5.4. *Suppose that for some $\alpha > 0$ (under the assumptions in Section 4) $v \in \mathcal{B}_\tau^{\alpha+1/d}(\Theta)$ with*

$$\frac{1}{\tau} = \alpha + \frac{1}{2}. \tag{5.29}$$

Then

$$\sigma_{N,H_0^1(\Omega)}(v) \lesssim \|v\|_{\mathcal{B}_\tau^{\alpha+1/d}(\Theta)} N^{-\alpha}, \tag{5.30}$$

with a constant depending only on $d, \mathbf{p}(\Theta), k$. Thus, whenever $\alpha + 1/d < k/d$, the Besov regularity $v \in B_\tau^{1+\alpha d}(L_\tau(\Omega))$ ensures a best N-term error decay rate of $N^{-\alpha}$.

Proof: Rearrange the terms $\{\||\eta_\lambda|^{-1/d} d_\lambda(v) F_\lambda\|_2\}$ in decreasing order according to their size

$$\||\eta_{\lambda_1}|^{-1/d} d_{\lambda_1}(v) F_{\lambda_1}\|_2 \geq \||\eta_{\lambda_2}|^{-1/d} d_{\lambda_2}(v) F_{\lambda_2}\|_2 \geq \cdots$$

and set $S_N := \sum_{j=1}^N d_{\lambda_j}(v) F_{\lambda_j}$. Then by Theorem 5.2 and Corollary 5.3 we obtain, on account of the well-known characterization $\mathcal{A}_{\ell_2}^\alpha = \ell_\tau^w$, $\frac{1}{\tau} = \alpha + \frac{1}{2}$,

$$\begin{aligned} \|v - S_N\|_{H^1} &\sim \Big\| \sum_{j=N+1}^\infty d_{\lambda_j}(v) F_{\lambda_j} \Big\|_{H^1} \lesssim \Big(\sum_{j=N+1}^\infty |\eta_{\lambda_j}|^{-2/d} \|d_{\lambda_j}(v) F_{\lambda_j}\|_2^2 \Big)^{1/2} \\ &\lesssim N^{-\alpha} \Big\| \{|\eta_\lambda|^{-1/d} \|d_\lambda(v) F_\lambda\|_2\} \Big\|_{\ell_\tau^w}, \end{aligned}$$

where for the decreasing rearrangement $(a_j^*)_{j \in \mathbb{N}}$ of the sequence $\mathbf{a} = (a_\lambda)_{\lambda \in \Lambda}$

$$\|\mathbf{a}\|_{\ell_\tau^w} := \sup_{n \in \mathbb{N}} n^{1/\tau} |a_n^*|.$$

Since $\|\mathbf{a}\|_{\ell_\tau^w} \lesssim \|\mathbf{a}\|_{\ell_\tau}$ we conclude that

$$\begin{aligned}
\|v - S_N\|_{H^1} &\lesssim N^{-\alpha}\left(\sum_{\lambda\in\Lambda}|\eta_\lambda|^{-\tau/d}\|d_\lambda(v)F_\lambda\|_2^\tau\right)^{1/\tau} \sim N^{-\alpha}\left(\sum_{\lambda\in\Lambda}|\eta_\lambda|^{-\frac{\tau}{d}}|\eta_\lambda|^{\frac{\tau}{2}-1}\|d_\lambda(v)F_\lambda\|_\tau^\tau\right)^{1/\tau}\\
&\sim N^{-\alpha}\left(\sum_{\lambda\in\Lambda}|\eta_\lambda|^{-\frac{\tau}{d}-\alpha\tau}\|d_\lambda(v)F_\lambda\|_\tau^\tau\right)^{1/\tau} = N^{-\alpha}\|v\|_{\mathcal{B}_\tau^{\alpha+1/d}(\Theta)},
\end{aligned}$$

where we have used (3.48) and (5.29). In view of Corollary 5.1, this completes the proof. □

The above assertion means that a proper placement of degrees of freedom preserves a best approximation rate in H^1 under the weakest excess smoothness of order $d\alpha$ that still ensures embedding in $H^1(\Omega)$.

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